

SPDES DRIVEN BY BOUNDARY LÉVY NOISE: UNIQUENESS OF THE INVARIANT MEASURE

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ABSTRACT. Let u be the solution to the following stochastic evolution equation

$$(1) \quad \begin{cases} du(t) &= Au(t) dt + B dL(t), \quad t > 0; \\ u(0) &= x \end{cases}$$

taking values in an Hilbert space H , where L is a \mathbb{R} valued Lévy process, $A : H \rightarrow H$ and $B : \mathbb{R} \rightarrow H$ possible unbounded operators. A typical example of such an equation is a stochastic Partial differential equation with boundary noise. Let $\mathcal{P} = (\mathcal{P}_t)_{t \geq 0}$ the corresponding Markovian semigroup. We investigate under which condition on the Lévy process L and on the operator A and B the solution of Equation (1) is asymptotically strong Feller. In addition we apply our result to the damped wave equation driven by Lévy boundary noise of finite variation and show that the corresponding Markovian semigroup admits a unique invariant measure.

1. INTRODUCTION

Regularity properties of the Markovian semigroups of stochastic processes play an important role in studying the long time behavior of the process. If the Markovian semigroup is strong Feller and satisfies an irreducibility property, then it admits a unique invariant measure. To relax these conditions Hairer and Mattingly introduced in [13] the so called asymptotic strong Feller property. In particular, they proved that for the uniqueness of the invariant measure it is sufficient the existence of the invariant measure, some nondengeneracy property and that the Markovian semigroup is asymptotically strong Feller. To present the aim of this paper, let H be a Hilbert space. Let u be the unique solution of the infinite dimensional system with Poissonian noise, formally written as

$$(2) \quad \begin{cases} du(t, x) &= Au(t, x) dt + \int_{\mathbb{R}} Bz \tilde{\eta}(dz, dt), \quad t > 0, \\ u(0, x) &= x. \end{cases}$$

In this equation, $A : H \rightarrow H$ is a linear operator generating a strongly continuous semigroup on H , $B : \mathbb{R} \rightarrow H$ is a certain mappings specified later and $\eta : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^+) \rightarrow \mathbb{N}_0 \cup \{\infty\}$ is a compensated Poisson random measure over a probability space $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and intensity measure ν . Let $\mathcal{P} = (\mathcal{P}_t)_{t \geq 0}$ be the Markovian semigroup induced on H , i.e.

$$(3) \quad \mathcal{P}_t \phi(x) := \mathbb{E} \phi(u(t, x)), \quad x \in H, \quad t > 0, \quad \phi \in C(H).$$

A typical example of such an equation is a stochastic partial differential equation with boundary noise. The aim of this paper is to verify under which conditions on A , B and η the Markovian semigroup generated by the solution of (2) admits an unique invariant measure.

Concerning the uniqueness of the invariant measure of SPDEs driven by Lévy processes only a few results exist. One of the first results in this direction were established in the articles of Chojnowska-Michalik [7, 8]. Next, Fournier [12] investigated SPDEs driven by space time Poissonian noise. Applebaum analysed in [3] the analytic property of the generalised Mehler semigroup induced by Lévy noise and in [2] the self-decomposability of a Lévy noise in Hilbert space. Further works are the two articles of Priola and Zabczyk [23, 24]. We also refer to [16], [25], [26] for some recent results and review of progress for the study of the ergodicity of the Markovian semigroup associated to the solution of a Lévy driven SPDEs.

The structure of the paper is the following. In Section 2 we give some preliminary notations and hypotheses used throughout the paper. In the same section we also proved an important relation between the asymptotic Feller property and null controllability with vanishing energy. Section 3 is devoted to the proof of the uniqueness of the invariant measure of the Markovian semigroup associated to the solution of (2). In fact, we established that if the Markovian semigroup satisfies the asymptotic strong feller property and that A and B generates an approximate null controllable system, then the aforementioned semigroup admits an unique invariant measure. We apply our results in Section 4 to a damped wave equations driven by boundary Lévy noise. The last part of the paper is some appendices collecting some technical results about the change of measure. The proofs of our results are a combination of the change of measure formula given by Bismuth, Graveriaux and Jacod [4] and Sato [27] (see also [14]) and the method used by Bohdan and Seidler [17].

Notation 1. Let $\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}_0^+ := (0, \infty)$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\bar{\mathbb{N}} := \mathbb{N}_0 \cup \{\infty\}$. Let (Z, \mathcal{Z}) be a measurable space. By $M_+(Z)$ we denote the family of all positive measures on Z , by $\mathcal{M}_+(Z)$ we denote the σ -field on $M_+(Z)$ generated by functions $i_B : M_+(Z) \ni \mu \mapsto \mu(B) \in \mathbb{R}_+$, $B \in \mathcal{Z}$. By $M_I(Z)$ we denote the family of all σ -finite integer valued measures on Z , by $\mathcal{M}_I(Z)$ we denote the σ -field on $M_I(Z)$ generated by functions $i_B : M_I(Z) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{Z}$. By $M_\sigma^+(Z)$ we denote the set of all σ -finite and positive measures on Z , by $\mathcal{M}_\sigma^+(Z)$ we denote the σ -field on $M_\sigma^+(Z)$ generated by functions $i_B : M_\sigma^+(Z) \ni \mu \mapsto \mu(B) \in \mathbb{R}$, $B \in \mathcal{Z}$. We denote by $B(Z)$ the set of all Borel measurable, real-valued, bounded functions.

For a Hilbert space H , by $C_b(H)$ the space of all uniformly continuous and bounded mappings $\phi : H \rightarrow \mathbb{R}$ endowed with the norm $|\phi|_\infty = \sup_{x \in H} |\phi(x)|$.

2. PRELIMINARIES

One way to handle Lévy processes is to work with the associated Poisson random measure. In this section we will define the setting in which the results can be formulated. We start with defining a time homogenous Poisson random measure.

Definition 2.1. Let (Z, \mathcal{Z}) be a measurable space and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. A time homogeneous Poisson random measure η on (Z, \mathcal{Z}) over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, is a measurable function $\eta : (\Omega, \mathcal{F}) \rightarrow (M_I(Z \times [0, \infty)), \mathcal{M}_I(Z \times [0, \infty)))$, such that

- (i) $\eta(\emptyset \times I) = 0$ a.s. for $I \in \mathcal{B}([0, \infty))$ and $\eta(A \times \emptyset) = 0$ a.s. for $A \in \mathcal{Z}$;
- (ii) for each $B \times I \in \mathcal{Z} \times \mathcal{B}([0, \infty))$, $\eta(B \times I) := i_{B \times I} \circ \eta : \Omega \rightarrow \bar{\mathbb{N}}$ is a Poisson random variable with parameter¹ $\nu(B)\lambda(I)$.

¹If $\nu(B)\lambda(I) = \infty$, then obviously $\eta(B \times I) = \infty$ a.s..

- (iii) η is independently scattered, i.e. if the sets $B_j \times I_j \in \mathcal{Z} \times \mathcal{B}([0, \infty))$, $j = 1, \dots, n$, are pairwise disjoint, then the random variables $\eta(B_j \times I_j)$, $j = 1, \dots, n$ are mutually independent.
- (iv) for each $U \in \mathcal{Z}$, the $\bar{\mathbb{N}}$ -valued process $(N(t, U))_{t \geq 0}$ defined by

$$N(t, U) := \eta(U \times (0, t]), \quad t \geq 0$$

is $(\mathcal{F}_t)_{t \geq 0}$ -adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent of \mathcal{F}_s .

The measure ν defined by

$$\nu : \mathcal{Z} \ni A \mapsto \mathbb{E}\eta(A \times (0, 1]) \in \bar{\mathbb{N}}$$

is called the intensity of η .

If the intensity of a Poisson random measure is a Lévy measure, then one can construct from the Poisson random measure a Lévy process. Vice versa, tracing the jumps, one can find a Poisson random measure associated to each Lévy process. For more details on this relationship we refer to [1, 5].

Let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability measure with right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, η be a time homogeneous Poisson random measure on \mathbb{R} over \mathfrak{A} with intensity ν being a Lévy measure² and compensator γ defined by

$$\gamma : \mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, \infty)) \ni (A \times I) \mapsto \gamma(A \times I) := \nu(A) \lambda(I) \in \mathbb{R}_0^+.$$

Hypothesis 1. We assume that the Lévy measure has a density k and there exist an index $\alpha \in (1, 2]$ and constants $K_0 > 0$ and $r_0 > 0$ such that

$$k(r) = K_0 |r|^{-\alpha-1}, \quad \text{for all } |r| \geq r_0.$$

Let H be a Hilbert space, $A : H \rightarrow H$ be a generator of a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ on H and $B : \mathbb{R} \rightarrow D(A^{-\gamma})$ for some $\gamma < \frac{1}{2}$. Let u be the solution of the following stochastic evolution system

$$(4) \quad \begin{cases} du(t, x) &= Au(t, x) + \int_{\mathbb{R}} Bz \tilde{\eta}(dz, ds), \\ u(0, x) &= x \in H. \end{cases}$$

Typical examples of such system are SPDEs with boundary noise and is presented in the following example (for more details we refer to section 4).

Example 2.2. We consider the vibration of a string of length 2π where one end is fixed and the other end is perturbed by a Levy noise. To be more precise, let $T > 0$ and $\alpha > 0$. We consider the system

$$(5) \quad \begin{cases} u_{tt}(t, \xi) - u_{\xi\xi}(t, \xi) + \alpha u(t, \xi) = 0, & t \in (0, T), \xi \in (0, 2\pi), \\ u(t, 0) = 0, & t \in (0, T), \\ u_{\xi}(t, 2\pi) = \dot{L}_t, & t \in (0, T), \\ u(0, \xi) = x_0(\xi), \quad u_t(0, \xi) = x_1(\xi), & \xi \in (0, 2\pi), \end{cases}$$

where \dot{L} is the Radon Nikodym derivative of a real valued Lévy process with intensity measure ν , $x_0 \in H_0^1(0, 2\pi)$ and $x_1 \in L^2(0, 2\pi)$. The existence of solution can be shown by standard fixed point arguments and can be found in [21] (see also [6]).

²A Lévy measure on \mathbb{R} is a σ -finite measure such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (|z|^2 \wedge 1) \nu(dz) < \infty$.

Under appropriate conditions on B , it can be shown that u admits a Markovian transition semigroup $(\mathcal{P}_t)_{t \geq 0}$ on H given by

$$(6) \quad \mathcal{P}_t \phi(x) := \mathbb{E} \phi(u(x, t)), \quad \phi \in C_b(H), \quad t \geq 0.$$

In addition, if the semigroup generated by A is of contractive type, i.e. there exists a $\omega > 0$ and $M > 0$ such that $\|e^{-tA}\|_{L(H,H)} \leq M e^{-\omega t}$, $t \geq 0$, then by direct calculations the following a priori estimate can be shown

$$(7) \quad \mathbb{E} |u(t, x)|_H^2 \leq e^{-\omega t} M \mathbb{E} |x|_H^2 + M C(t, \nu) \int_{\mathbb{R}} |z|^2 \nu(dz)$$

where $C(t, \nu) = \int_0^t e^{-\gamma s} s^{-2\gamma} ds$. Note that $\lim_{t \rightarrow \infty} C(t, \omega) < \infty$. Now, the existence of the invariant measure can be shown by an application of the Krylov-Bogoliubov Theorem. In Theorem 2.6 we verify some conditions under which the semigroup is asymptotically strong Feller, and, in Theorem 3.1 we show under which conditions the semigroup is non degenerate. From these two theorems follows the uniqueness of the invariant measure. In section 4 we apply the results to the damped wave equation described in Example 2.2.

Remark 1. *It can be shown that the damped wave equation with boundary noise has an invariant measure, for details we refer to section 4.*

Before continuing we would like to introduce some definitions from control theory. Again, H denotes a Hilbert space, $A : H \rightarrow H$ a generator of a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ on H and $B : \mathbb{R} \rightarrow H$. Then we say that the system

$$(8) \quad \begin{cases} \dot{u}^c(t, x, v) &= A u^c(t, x, v) + B v(t), \quad t \geq 0, \\ u^c(0, x, v) &= x, \end{cases}$$

is *null controllable*, iff for any $x \in H$ there exists a $v \in L^2([0, \infty); \mathbb{R})$ and a time $t > 0$ such that $u^c(t, x, v) = 0$. Furthermore, system (8) is *null controllable with vanishing energy* (see [20, 22]), if it is null controllable and for any $x \in H$ there exists a sequence of controls $\{v_n : n \in \mathbb{N}\} \subset L^2([0, \infty); \mathbb{R})$ and of times $\{t_n \geq 0 : n \in \mathbb{N}\}$ such that $u(t_n, x, v_n) = 0$ for any $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \int_0^{t_n} |v_n(s)|_{\mathbb{R}}^2 ds = 0.$$

We say that the system (8) is *approximate controllable* in $x \in H$ if for each $y \in H$ and $\epsilon > 0$ there exists a time $t > 0$ and a control $v \in L^2([0, \infty); \mathbb{R})$ such that

$$|u^c(t, x, v) - y| \leq \epsilon.$$

Now, it may happen that one has to bound a higher order norm of the control. Therefore, fix $q \geq 2$. Now, we call a system (8) *approximate controllable in $x \in H$ in L^q -mean* if for each $y \in H$ and $\epsilon > 0$ there exists a time $t > 0$ and a control $v \in L^q([0, \infty); \mathbb{R})$ such that

$$|u^c(t, x, v) - y| \leq \epsilon.$$

Now we are ready to present our result about the asymptotically strong Feller property. However, before introducing the definition of asymptotic strong Feller, let us introduce some notation.

Definition 2.3. Let \mathcal{X} be a metrizable space. A function $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ is called a pseudo metric, if

- (1) $x = y \Rightarrow d(x, y) = 0$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
- (3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \mathcal{X}$.

Let d_1 and d_2 be two pseudo-metrics on \mathcal{X} , d_2 is larger than d_1 , if $d_2(x, y) \geq d_1(x, y)$ for all $x, y \in \mathcal{X}$. A sequence $\{d_n : n \in \mathbb{N}\}$ of pseudo-metrics is called increasing, if d_{n+1} is larger than d_n for all $n \in \mathbb{N}$.

Definition 2.4. An increasing sequence $\{d_n : n \in \mathbb{N}\}$ of pseudo metrics is called a totally separating system of pseudo metrics for \mathcal{X} , if $\lim_{n \rightarrow \infty} d_n(x, y) = 1$ for all $x, y \in \mathcal{X}$, $x \neq y$.

In order to define the distance of two probability measures on \mathcal{X} let us introduce the following metric. Let d be a pseudo-metric on \mathcal{X} and $\phi : \mathcal{X} \rightarrow \mathbb{R}$, ϕ Lipschitz continuous. Then, we define

$$\|\phi\|_d := \sup_{\substack{x, y \in \mathcal{X} \\ x \neq y}} \frac{|\phi(x) - \phi(y)|}{d(x, y)},$$

and let $L(\mathcal{X}, d) := \{\phi : \mathcal{X} \rightarrow \mathbb{R}; \phi \text{ Lipschitz continuous and } \|\phi\|_d < \infty\}$. Let \mathcal{P}_1 and \mathcal{P}_2 two probability measures on \mathcal{X} . Then

$$\|\mathcal{P}_1 - \mathcal{P}_2\|_d := \sup_{\substack{\phi \in L(\mathcal{X}, d) \\ \|\phi\|_d = 1}} \int_{\mathcal{X}} \phi(x) (\mathcal{P}_1 - \mathcal{P}_2)(dx).$$

Definition 2.5. Let \mathcal{X} be a Polish space. A Markovian transition semigroup $\mathcal{P} = (\mathcal{P}_t)_{t \geq 0}$ is called asymptotically strong Feller in $x \in \mathcal{X}$, if there exists a totally separating system of pseudo-metrics $\{d_n : n \in \mathbb{N}\}$ on \mathcal{X} and a non decreasing sequence $\{t_n : n \in \mathbb{N}\}$ such that

$$\inf_{U \in \mathbb{R}_x} \limsup_{n \rightarrow \infty} \sup_{y \in U} \|\mathcal{P}_{t_n}(x, \cdot) - \mathcal{P}_{t_n}(y, \cdot)\|_{d_n} = 0.$$

Now, we are ready to introduce the following Theorem.

Theorem 2.6. If $\alpha > 1$, and if the system (8) is null controllable with vanishing energy, then the Markovian semigroup of system (4) is asymptotically strong Feller.

Proof of Theorem 2.6. : We will switch for technical reasons to another representation of the Poisson random measure. Let $\overline{\mathfrak{A}} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ be a filtered probability space with filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t \geq 0}$ and let μ be a Poisson random measure on \mathbb{R} over $\overline{\mathfrak{A}}$ having intensity measure λ (Lebesgue measure). The compensator of μ is denoted by γ and given by

$$\mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, \infty)) \ni A \times I \mapsto \gamma(A \times I) := \lambda(A) \lambda(I).$$

Let

$$(9) \quad c : \mathbb{R}^+ \ni r \mapsto \sup_{\rho > 0} \left\{ \int_{\rho}^{\infty} k(s) ds \geq r \right\} \quad \text{if } r > 0.$$

Remark 2. Observe that Hypothesis 1 implies that there exists a number $r_1 > 0$ and a constant δ_0 such that for all $r \geq r_1$

$$(10) \quad c(r) = \delta_0 r^{-\frac{1}{\alpha}}, \quad \text{and} \quad c(-r) = -\delta_0 r^{-\frac{1}{\alpha}}.$$

A short calculation shows that the distributions of $L = \{L(t) : 0 \leq t < \infty\}$ and $L^c = \{L^c(t) : 0 \leq t < \infty\}$ are equal, where

$$L(t) := \int_0^t \int_{\mathbb{R}} z \tilde{\eta}(dz, ds), \quad t \geq 0,$$

and

$$L^c(t) := \int_0^t \int_{\mathbb{R}} c(z) \tilde{\mu}(dz, ds), \quad t \geq 0.$$

Now, the stochastic evolution equation given in (4) reads as follows

$$(11) \quad \begin{cases} du(t, x) &= Au(t, x) + \int_{\mathbb{R}} Bc(z) \tilde{\mu}(dz, ds), \\ u(0, x) &= x \in H. \end{cases}$$

In order to show the asymptotical strong Feller property, we have to show that there exist an increasing sequence $\{t_n : n \in \mathbb{N}\}$ and a totally separating sequence of pseudometrics $\{d_n : n \in \mathbb{N}\}$ such that

$$(12) \quad \lim_{\epsilon > 0} \limsup_{n \rightarrow \infty} \sup_{y \in B(x, \epsilon)} \|\mathcal{P}_{t_n}(x, \cdot) - \mathcal{P}_{t_n}(y, \cdot)\|_{d_n} = 0.$$

Fix $\epsilon > 0$ and let $\{a_n : n \in \mathbb{N}\}$ be a sequence of positive real numbers converging to zero. Let

$$d_n(x, y) := 1 \wedge (|x - y|_H / a_n), \quad x, y \in H, \quad n \in \mathbb{N}.$$

Remark 3. *The sequence $\{d_n : n \in \mathbb{N}\}$ is a totally separating sequence of pseudometrics.*

Let $\phi \in C_b(H)$. Then there exists a sequence $\{\phi_n : n \in \mathbb{N}\}$, $\phi_n \in C_b^\infty(H)$, such that $\phi_n \rightarrow \phi$ pointwise, $\|\phi_n\|_\infty \leq \|\phi\|_\infty$ and $\|\phi_n\|_{d_n} \leq 1$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Then we have to find a sequence of times $\{t_n : n \in \mathbb{N}\}$ such that (12) holds. Therefore, put $y - x = \epsilon a$. By the assumption, there exist for any $n \in \mathbb{N}$ a control v^n and a time $t_n > 0$ such that $a_n^2 \geq \int_0^{t_n} |v^n(s)|^2 ds$ and the solution to

$$\begin{cases} du(t, x, v^n) &= Au(t, x, v^n) dt + Bv^n(t) dt, \quad 0 \leq t \leq t_n \\ u(0, x, v^n) &= x, \end{cases}$$

satisfy $u(t_n, x, v^n) = u(t_n, x + a, 0)$. Put $v_\epsilon^n := \epsilon \cdot v^n$. Then we have $\int_0^{t_n} |v_\epsilon^n(s)|^2 ds = \epsilon^2 a_n^2$ and by the linearity of A and B , $u(t_n, x, v_\epsilon^n) = u(t_n, x + \epsilon a, 0)$. In the following lines we will give an estimate of

$$(13) \quad \|\mathcal{P}_{t_n}(x, \cdot) - \mathcal{P}_{t_n}(y, \cdot)\|_{d_n} = \left| \mathbb{E}^\mathbb{P} [\phi_n(u(t_n, x + \epsilon a))] - \mathbb{E}^\mathbb{P} [\phi_n(u(t_n, x))] \right|.$$

Next, let us introduce a transformation $\theta_n^\epsilon : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$, such that we have

$$\int_{\mathbb{R}} (c(z) - c(\theta_n^\epsilon(s, z))) \lambda(dz) = v_\epsilon^n(s) \quad \text{for all } s \in [0, t_n].$$

In fact, by Lemma B.1 we can suppose that such a transformation θ_n^ϵ exists and is given by

$$(14) \quad \theta_n^\epsilon : [0, \infty) \times \mathbb{R} \ni (s, z) \mapsto z + \rho(K(|v_\epsilon^n(s)|), z) \operatorname{sgn}(v_\epsilon^n(s)),$$

where K is the inverse of (47). In addition, let μ_ϵ^n be a random measure defined by

$$(15) \quad \mathcal{B}(\mathcal{Z}) \times \mathcal{B}([0, t]) \ni A \times I \mapsto \mu_\epsilon^n(A \times I) = \int_I \int_A 1_A(\theta_n^\epsilon(s, z)) \mu(dz, ds)$$

Let $\mathbb{Q}^{\epsilon,n}$ be that probability measure on $\bar{\mathfrak{A}}$ such that μ_ϵ^n has compensator λ . Let u_ϵ^n be the solution to

$$\begin{cases} du_\epsilon^n(t, x) &= Au_\epsilon^n(t, x) dt + \int_{\mathbb{R}} B[c(\theta_\epsilon^n(t, z)) - c(z)]\mu(dz, dt) + \int_{\mathbb{R}} Bc(z)(\mu - \gamma)(dz, dt), \\ u_\epsilon^n(0, x) &= x, \end{cases}$$

and let $u_{\mu,n,\epsilon}^c$ be solution to

$$\begin{cases} du_{\mu,n,\epsilon}^c(t, x, v_\epsilon^n) &= Au_{\mu,n,\epsilon}^c(t, x, v_\epsilon^n) dt + Bv_\epsilon^n(t) dt + \int_{\mathbb{R}} Bc(z)(\mu - \gamma)(dz, dt), \\ u_{\mu,n,\epsilon}^c(0, x, v_\epsilon^n) &= x. \end{cases}$$

Observe that, first, by the choice of the transformation θ_ϵ^n under $\mathbb{Q}^{\epsilon,n}$ the random variable $u_\epsilon^n(t_n, x)$ is identical in law to the process $u(t_n, x)$. In particular, we have

$$\mathbb{E}^{\mathbb{Q}_{t_n}^{\epsilon,n}} [\phi_n(u_\epsilon^n(t_n, x))] = \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x))].$$

Secondly, by the choice of the control and the linearity of A we have $u_{\mu,n,\epsilon}^c(t_n, x, v_\epsilon^n) = u(t_n, x + \epsilon a)$. For $t \geq 0$ let $\mathbb{Q}_t^{\epsilon,n}$ be the restriction of $\mathbb{Q}^{\epsilon,n}$ onto $\bar{\mathcal{F}}_t$ and $\bar{\mathbb{P}}_t$ be the restriction of $\bar{\mathbb{P}}$ onto $\bar{\mathcal{F}}_t$. Now, we can write

$$\begin{aligned} & \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x + \epsilon a))] - \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x))] \\ &= \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u_{\mu,n,\epsilon}^c(t_n, x, v_\epsilon^n))] - \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u_\epsilon^n(t_n, x))] + \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u_\epsilon^n(t_n, x))] - \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x))] \\ &= \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u_{\mu,n,\epsilon}^c(t_n, x, v_\epsilon^n)) - \phi_n(u_\epsilon^n(t_n, x))] + \mathbb{E}^{\bar{\mathbb{P}}} \left[\left[1 - \frac{d\mathbb{Q}_{t_n}^{\epsilon,n}}{d\bar{\mathbb{P}}_{t_n}} \right] \phi_n(u_\epsilon^n(t_n, x)) \right] \\ & \quad + \mathbb{E}^{\mathbb{Q}_{t_n}^{\epsilon,n}} [\phi_n(u_\epsilon^n(t_n, x))] - \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x))] \\ &= \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u_{\mu,n,\epsilon}^c(t_n, x, v_\epsilon^n)) - \phi_n(u_\epsilon^n(t_n, x))] + \mathbb{E}^{\bar{\mathbb{P}}} \left[\left[1 - \frac{d\mathbb{Q}_{t_n}^{\epsilon,n}}{d\bar{\mathbb{P}}_{t_n}} \right] \phi_n(u_\epsilon^n(t_n, x)) \right]. \end{aligned}$$

Next,

$$\begin{aligned} & \left| \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x + \epsilon a))] - \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x))] \right| \\ & \leq \frac{1}{a_n} \mathbb{E}^{\bar{\mathbb{P}}} |u_{\mu,n,\epsilon}^c(t_n, x, v_\epsilon^n) - u_\epsilon^n(t_n, x)| + \|\phi_n\|_\infty \mathbb{E}^{\bar{\mathbb{P}}} \left| 1 - \frac{d\mathbb{Q}_{t_n}^{\epsilon,n}}{d\bar{\mathbb{P}}_{t_n}} \right|. \end{aligned}$$

Let us put

$$I_1^\epsilon := \mathbb{E}^{\bar{\mathbb{P}}} |u_{\mu,n,\epsilon}^c(t_n, x, v_\epsilon^n) - u_\epsilon^n(t_n, x)|, \quad \text{and} \quad I_2^\epsilon := \mathbb{E}^{\bar{\mathbb{P}}} \left| 1 - \frac{d\mathbb{Q}_{t_n}^{\epsilon,n}}{d\bar{\mathbb{P}}_{t_n}} \right|.$$

Next, by the construction of $u_\epsilon^n(t, x)$ and $u_{\mu,n,\epsilon}^c(t, x, v_\epsilon^n)$ we see that

$$u_\epsilon^n(t_n, x) - u_{\mu,n,\epsilon}^c(t_n, x, v_\epsilon^n) = \int_0^{t_n} \int_{\mathbb{R}} e^{-(t_n-s)A} B [c(z) - c(\theta_\epsilon^n(s, z))] (\gamma - \mu)(dz, ds).$$

and therefore

$$\begin{aligned} & \mathbb{E}^{\bar{\mathbb{P}}} |u_\epsilon^n(t_n, x) - u_{\mu,n,\epsilon}^c(t_n, x, v_\epsilon^n)|^2 \\ & \leq \mathbb{E}^{\bar{\mathbb{P}}} \int_0^{t_n} \int_{\mathbb{R}} \left| e^{-(t_n-s)A} B [c(z) - c(\theta_\epsilon^n(s, z))] \right|^2 \lambda(dz) ds \\ & \leq \mathbb{E}^{\bar{\mathbb{P}}} \int_0^{t_n} \left| e^{-(t_n-s)A} B v_\epsilon^n(s) \right|^2 ds. \end{aligned}$$

and, hence,

$$\begin{aligned}
|I_1^\epsilon| &\leq \mathbb{E}^{\mathbb{P}} |u(t_n, x) - u_{\mu, n, \epsilon}^c(t_n, x, v_\epsilon^n)|^2 \leq \mathbb{E}^{\mathbb{P}} \int_0^{t_n} \left| e^{-(t_n-s)A} B v_\epsilon^n(s) \right|^2 ds \\
&\leq \left(\int_0^{t_n} (t_n - s)^{-2\gamma} e^{-2(t_n-s)\rho} ds \right)^{\frac{1}{2}} \left(\int_0^{t_n} |B v_\epsilon^n(s)|_{D(A^{-\gamma})}^2 ds \right)^{\frac{1}{2}} \\
&\leq C(\gamma, \rho) \left(\int_0^{t_n} |v_\epsilon^n(s)|_{\mathbb{R}}^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

To give an estimate of the second term I_2^ϵ we apply [14, Theorem 1] to get an exact representation of the Radon Nikodym derivative. In particular, we obtain

$$\begin{aligned}
I_2^\epsilon &\leq \mathbb{E}^{\mathbb{P}} \left[\left(1 - \frac{d\mathbb{Q}_{t_n}^{\epsilon, n}}{d\mathbb{P}_{t_n}} \right) \phi(u_\epsilon^n(t_n, x)) \right] \\
&\leq \mathbb{E}^{\mathbb{P}} [(1 - \mathcal{G}_\epsilon^n(t_n)) \phi(u_\epsilon^n(t_n, x))]
\end{aligned}$$

where \mathcal{G}_ϵ^n is defined by (see Lemma C.1 and (14))

(16)

$$\begin{cases} d\mathcal{G}_\epsilon^n(t) &= \mathcal{G}_\epsilon^n(t) (\rho_z(K(|v_\epsilon^n(s)|)z) \operatorname{sgn}(v_\epsilon^n(s))) (\mu - \gamma)(dz, ds) \\ \mathcal{G}_\epsilon^n(0) &= 1. \end{cases}$$

Now, the Hölder inequality gives

$$I_2^\epsilon \leq \mathbb{E}^{\mathbb{P}} |1 - \mathcal{G}_\epsilon^n(t_n)| |\phi|_\infty.$$

First we will give an estimate of $\mathbb{E} \sup_{0 \leq s \leq t_n} |\mathcal{G}_\epsilon^n(s)|$. An application of the Itô formula and the estimate (48) give for $0 \leq t \leq t_n$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}} \sup_{0 \leq s \leq t} |\mathcal{G}_\epsilon^n(s)| &\leq 1 + \int_0^t \int_{\mathbb{R}^+} |\mathcal{G}_\epsilon^n(s-)| |K(v_\epsilon^n(s))| |\rho_z(K(v_\epsilon^n(s)), z)| dz ds \\
(17) \qquad \qquad \qquad &\leq 1 + C \int_0^t |\mathcal{G}_\epsilon^n(s-)| |K(|v_\epsilon^n(s)|)| \int_{\mathbb{R}^+} |\rho_z(K(v_\epsilon^n(s)), z)| dz ds.
\end{aligned}$$

By Corollary B.3 it follows

$$\int_{\mathbb{R}^+} |\rho_z(K(|v_\epsilon^n(s)|), z)| dz \leq C(r_1) |v_\epsilon^n(s)|^2$$

Substituting (18) in (17) we obtain

$$(18) \qquad \mathbb{E}^{\mathbb{P}} \sup_{0 \leq s \leq t} |\mathcal{G}_\epsilon^n(s)| \leq 1 + C(r_1) \int_0^t |\mathcal{G}_\epsilon^n(s-)| |v_\epsilon^n(s)|^2 ds$$

$$(19) \qquad \qquad \qquad \leq 1 + C(r_1) \mathbb{E} \sup_{0 \leq s \leq t} |\mathcal{G}_\epsilon^n(s)| \int_0^t |v_\epsilon^n(s)|^2 ds.$$

Since $\int_0^{t_n} |v_\epsilon^n(s)|^2 ds \leq a_n^2$ and $a_n \rightarrow 0$, there exists a $n_0 \in \mathbb{N}$ such that $C(r_1) a_n^2 < 1/2$. Therefore, for $n \geq n_0$ we obtain

$$\mathbb{E}^{\mathbb{P}} \sup_{0 \leq s \leq t} |\mathcal{G}_\epsilon^n(s)| \leq 2.$$

Again applying the Itô formula and the considerations above we obtain

$$\begin{aligned} \mathbb{E}^{\bar{\mathbb{P}}} |\mathcal{G}_\epsilon^n(t_n) - 1| &\leq C(r_1) \mathbb{E}^{\bar{\mathbb{P}}} \int_0^{t_n} |\mathcal{G}_\epsilon^n(s-)| |v_\epsilon^n(s)|^2 ds \\ &\leq C(r_1) \mathbb{E}^{\bar{\mathbb{P}}} \sup_{0 \leq s \leq t_n} |\mathcal{G}_\epsilon^n(s)| \int_0^{t_n} |v_\epsilon^n(s)|^2 ds, \\ &\leq C(r_1) 2\epsilon a_n^2. \end{aligned}$$

Going back to Ansatz (13) and taking the limit, it follows that there exists some constants $C_1, C_2 > 0$ and some $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$

$$\begin{aligned} &\left| \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x + \epsilon a))] - \mathbb{E}^{\bar{\mathbb{P}}} [\phi_n(u(t_n, x))] \right| \\ &\leq \left\{ \frac{C_1}{a_n} \left(\int_0^{t_n} |v_\epsilon^n(s)|^2 ds \right)^{\frac{1}{2}} + C_2 \|\phi\|_\infty 2\epsilon^2 a_n^2 \right\}. \end{aligned}$$

Hence, we have

$$\leq \left\{ \frac{C_1 \epsilon a_n}{a_n} + C_2 \epsilon a_n \right\}.$$

Taking the limit $n \rightarrow \infty$ we get

$$\limsup_{n \rightarrow \infty} \sup_{y \in B(x, \epsilon)} \|\mathcal{P}_{t_n}(x, \cdot) - \mathcal{P}_{t_n}(y, \cdot)\|_{d_n} \leq C\epsilon.$$

Taking the limit $\epsilon \rightarrow 0$, the assertion follows.

3. UNIQUENESS OF THE INVARIANT MEASURE

By [13, Corollary 3.17] we know that if the semigroup is asymptotically strong Feller and there exists a point $x \in H$ such that $x \in \text{supp}(\sigma)$, whenever σ is an invariant measure of the Markovian semigroup $(\mathcal{P}_t)_{t \geq 0}$, then the semigroup $(\mathcal{P}_t)_{t \geq 0}$ admits at most one invariant measure.

Assume for the time being that the semigroup $(\mathcal{P}_t)_{t \geq 0}$ is asymptotically strong Feller. Hence, it remains to prove that there exists a $x \in H$ such that $x \in \text{supp}(\sigma)$. Now, the two following properties imply that $0 \in \text{supp}(\sigma)$ ³ whenever σ is an invariant measure.

- There exists a constant $C > 0$ such that

$$(20) \quad \inf_{\{\rho \text{ is an invariant measure}\}} \rho(B_H(C)) > 0.$$

- For all $\delta > 0$ and for all $x \in B_H(C)$ there exists a time $T_\delta > 0$ and some $\kappa > 0$ such that

$$(21) \quad \mathbb{P}(u(T_\delta, x) \in \mathcal{B}_H(\delta)) \geq \kappa.$$

It follows that $0 \in \text{supp}(\sigma)$ by the following observations. First, since σ is invariant we have

$$\sigma(\mathcal{B}_H(\delta)) \geq \kappa \cdot \inf_{\{\rho \text{ is invariant measure}\}} \rho(\bar{\mathcal{B}}_H(C)).$$

Now, estimates (20) and (21) give the assertion.

³Note, that $x \in \text{supp}(\sigma)$ iff for all $\delta > 0$, $\sigma(\mathcal{B}_H(\delta)) > 0$.

Hence, in order to show that system (4) admits an unique invariant measure, we have to verify estimate (20) and estimate (21). Estimate (20) follows by the fact that for any invariant measure σ of the semigroup $(\mathcal{P}_t)_{t \geq 0}$ there exists a constant $C > 0$ such that

$$(22) \quad \int_H |u|_H^2 \sigma(du) \leq C.$$

Estimate (22) follows by the same lines as in [18, Lemma B.1] and the a priori estimate (7). Now, an application of the Chebyscheff inequality leads to (20).

In the following lemma we prove under which controllability conditions on system (8) and hypothesis on the Lévy process, property (21) holds for system (4).

Lemma 3.1. *Assume that the system (8) is approximate null controllable and that Hypothesis 1 is satisfied. Then for all $\delta > 0$ and for all $x \in B_H(C)$ there exists a time $T_\delta > 0$ and a $\kappa > 0$ such that*

$$(23) \quad \mathbb{P}(u(T_\delta, x) \in B_H(\delta)) \geq \kappa.$$

Proof of Lemma 3.1. Again, like in the proof of Theorem 2.6 at page 5 we change the representation of the Poisson random measure. Therefore, let $\overline{\mathfrak{A}} = (\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}}, \overline{\mathbb{P}})$ be a filtered probability space and let μ be a Poisson random measure on \mathbb{R} over $\overline{\mathfrak{A}}$ having intensity measure λ . Let $c : \mathbb{R} \rightarrow \mathbb{R}$ be defined as in (9).

Fix $\delta > 0$, $T > 0$ and $x \in B_H(C)$. In order to prove Lemma 3.1 we need a result from control theory. Given $v \in L^2([0, \infty); \mathbb{R})$, let u^c be the solution to (see system (8))

$$(24) \quad \begin{cases} du^c(t, x, v) &= Au^c(t, x, v)dt + Bv(t)dt, \quad t \geq 0, \\ u^c(0, x, v) &= x. \end{cases}$$

Since the system (24) is approximate null controllable, there exists $v_0 \in L^2([0, \infty); \mathbb{R})$ such that

$$(25) \quad |u^c(T, x, v)| \leq \frac{\delta}{3}.$$

Choose $R \geq r_1$ such that

$$\left(\frac{3}{\delta}\right)^2 CTR^{1-\frac{2}{\alpha}} \leq \frac{1}{2},$$

(here, C is a generic constant, not depending on δ , T and R , see (31)) and put

$$(26) \quad g_R = \int_{B_{\mathbb{R}}(R)} c(z) \lambda(dz).$$

Let $\theta : \Omega \times [0, T]$ be a predictable transformation of \mathbb{R} such that

$$v(s) + g_R = \int_{\mathbb{R} \setminus B_{\mathbb{R}}(\rho)} [c(z) - c(\theta(s, z))] \lambda(dz), \quad s \in [0, T].$$

The existence of such a transformation is given by Lemma B.3. Let μ_θ the following random measure defined by

$$\mu_\theta : \mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, T]) \ni (A \times I) \mapsto \int_{\mathbb{R}^2} \int_I 1_A(\theta(s, z)) \mu(dz, ds).$$

Let \mathbb{Q} be the probability measure on \mathfrak{A} such that μ_θ has compensator γ . Then, the process u_μ^θ defined by

$$(27) \quad \begin{cases} du_\mu^\theta(t, x) &= Au_\mu^\theta(t, x)dt + \int_{\mathbb{R}} Bc(z) (\mu_\theta - \gamma)(dz, dt) \\ u_\mu^\theta(0, x) &= x. \end{cases}$$

has under \mathbb{Q} the same law as u , in particular

$$\mathbb{E}^{\mathbb{P}} [1_{[0, \delta]}(|u(T, x_0)|)] = \mathbb{E}^{\mathbb{Q}} [1_{[0, \delta]}(|u_\mu^\theta(T, x_0)|)].$$

Due to Lemma C.1 the density process $\mathcal{G}_\theta(t) = \frac{d\mathbb{Q}_t^\theta}{d\mathbb{P}_t}$ satisfy the following stochastic differential equation (see also page 8)

$$\begin{cases} d\mathcal{G}_\theta(t) &= \mathcal{G}_\theta(t) (\rho_z(K(|v(s)|), z)) \operatorname{sgn}(v(s)) (\mu - \gamma)(dz, ds) \\ \mathcal{G}_\theta(0) &= 1. \end{cases}$$

By the same calculation as in (17) we obtain for $0 \leq t \leq T$

$$\mathbb{E}^{\bar{\mathbb{P}}} \sup_{0 \leq s \leq t} |\mathcal{G}_\theta(s)| \leq 1 + \mathbb{E}^{\bar{\mathbb{P}}} \int_0^t \int_{\mathbb{R}^+} |\mathcal{G}_\theta(s-)| |\rho_z(K(|v(s)|), z)| dz ds.$$

In fact, one can even show for $0 \leq t \leq T$

$$\mathbb{E}^{\bar{\mathbb{P}}} \sup_{r \leq s \leq t} |\mathcal{G}_\theta(s)| \leq \mathbb{E}^{\bar{\mathbb{P}}} \sup_{0 \leq s \leq r} |\mathcal{G}_\theta(s)| + C(r_1) \mathbb{E}^{\bar{\mathbb{P}}} \left[\sup_{r \leq s \leq t} |\mathcal{G}_\theta(s)| \right] \left(\int_r^t |v(s)|^2 ds \right).$$

Since $v \in L^2([0, T]; \mathbb{R})$, there exists a partition $\{t_j : 1 \leq j \leq N\}$ of $[0, T]$ with

$$\int_{t_{j-1}}^{t_j} |v(s)|^2 ds < \frac{1}{2C}, \quad j = 2, \dots, N.$$

Therefore,

$$\mathbb{E}^{\bar{\mathbb{P}}} \sup_{0 \leq s \leq t_j} |\mathcal{G}_\theta(s)| \leq C_{j-1} + \frac{1}{2}$$

and we can conclude that there exists a constant $C(r_1) > 0$ depending on v such that

$$(28) \quad \mathbb{E}^{\bar{\mathbb{P}}} \sup_{0 \leq s \leq T} |\mathcal{G}_\theta(s)| \leq C(r_1).$$

In particular $C = N \cdot \frac{1}{2}$.

On the other hand we know that under $\bar{\mathbb{P}}$ the process u_μ^θ follows the following differential equation

$$(29) \quad \begin{cases} du_\mu^\theta(t, x) &= Au_\mu^\theta(t, x)dt + \int_{\mathbb{R}} B [c(\theta(t, z)) - c(z)] (\mu - \gamma)(dz, dt) \\ &\quad + \int_{\mathbb{R}} B [c(\theta(t, z)) - c(z)] \gamma_\sigma(dz, dt) + \int_{\mathbb{R}} Bc(z)(\mu - \gamma)(dz, dt), \\ u_\mu^\theta(0, x) &= x. \end{cases}$$

Hence we can write

$$\bar{\mathbb{P}}(|u(T, x)| \leq \delta) = \mathbb{Q}(|u_\mu^\theta(T, x)| \leq \delta) = \mathbb{E}^{\mathbb{Q}} [1_{|u_\mu^\theta(T, x)| \leq \delta}] = \mathbb{E}^{\bar{\mathbb{P}}} [\mathcal{G}_\theta(T) 1_{|u_\mu^\theta(T, x)| \leq \delta}].$$

By the inverse Hölder inequality we get

$$\bar{\mathbb{P}}(|u(T, x)| \leq \delta) \geq \frac{\mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|u_{\mu}^{\theta}(T, x)| \leq \delta} \right]}{\mathbb{E}^{\bar{\mathbb{P}}} [\mathcal{G}_{\theta}(T)]}.$$

The denominator, i.e. $\mathbb{E}^{\bar{\mathbb{P}}} [|\mathcal{G}_{\theta}(T)|]$, is bounded. In particular, we have by (28) and Hypothesis 1 that

$$\mathbb{E}^{\bar{\mathbb{P}}} [|\mathcal{G}_{\theta}(T)|] := r_1^{\frac{\alpha\beta_1+1}{\alpha}} \int_0^T |v(s)|^2 ds.$$

Next, we handle the numerator. Observe that by (25)

$$\begin{aligned} & \mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|u_{\mu}^{\theta}(T, x)| \leq \delta} \right] \\ (30) \quad & \geq \mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|u^c(T, x, v)| \leq \frac{\delta}{3}} 1_{|u^c(T, x, v) - u_{\mu}^{\theta}(T, x)| \leq \frac{\delta}{3}} \right] = \mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|u^c(T, x, v) - u_{\mu}^{\theta}(T, x)| \leq \frac{\delta}{3}} \right]. \end{aligned}$$

Rewriting the difference $\Delta(T) = u^c(T, x, v) - u_{\mu}^{\theta}(T, x)$ as follows

$$\begin{aligned} \Delta(T) &= \int_0^T \int_{\mathbb{R}} e^{-(t-s)A} B [c(\theta(t, z)) - c(z)] (\mu - \gamma)(dz, ds) \\ &\quad + \int_0^T \int_{\mathbb{R}} e^{-(t-s)A} B c(z) (\mu - \gamma)(dz, ds) + \int_0^T e^{-(t-s)A} B g(s) ds \\ &= \int_0^T \int_{\mathbb{R}} e^{-(t-s)A} B c(\theta(t, z)) (\mu - \gamma)(dz, ds) \\ &\quad + \int_0^T e^{-(t-s)A} B g(s) ds. \end{aligned}$$

we have

$$\mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|u_{\mu}^{\theta}(T, x)| \leq \delta} \right] \geq \mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|\Delta(T)| \leq \frac{\delta}{3}} \right].$$

To give a lower estimate of $\bar{\mathbb{P}}(|\Delta(T)| \leq \frac{\delta}{3})$ we apply the Bayes Theorem and get

$$\begin{aligned}
& \bar{\mathbb{P}}\left(|\Delta(T)| \leq \frac{\delta}{3}\right) = \bar{\mathbb{P}}(\mu(B_{\mathbb{R}}(R) \times [0, T]) = 0) \\
& \times \bar{\mathbb{P}}\left(\left|\int_0^{T_\rho} \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right. \right. \\
& \quad \left. \left. + \underbrace{\int_0^{T_\rho} \int_{B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds)}_{=\int_0^{T_\rho} e^{-(t-s)A} Bg(s) ds} \right. \right. \\
& \quad \left. \left. - \int_0^{T_\rho} e^{-(t-s)A} Bg(s) ds \right| \leq \frac{\delta}{3} \mid \mu(B_{\mathbb{R}}(R) \times [0, T]) = 0\right) \\
& + \bar{\mathbb{P}}(\mu(B_{\mathbb{R}}(R) \times [0, T]) > 0) \\
& \times \bar{\mathbb{P}}\left(\left|\int_0^{T_\rho} \int_{\mathbb{R}} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right. \right. \\
& \quad \left. \left. - \int_0^{T_\rho} e^{-(t-s)A} Bg(s) ds \right| \leq \frac{\delta}{3} \mid \mu(B_{\mathbb{R}}(R) \times [0, T]) > 0\right) \\
& \geq \bar{\mathbb{P}}(\mu(B_{\mathbb{R}}(R) \times [0, T]) = 0) \\
& \times \bar{\mathbb{P}}\left(\left|\int_0^{T_\rho} \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right| \leq \frac{\delta}{3} \mid \mu(B_{\mathbb{R}}(R) \times [0, T]) = 0\right).
\end{aligned}$$

By the Chebyscheff inequality we know that

$$\begin{aligned}
& \bar{\mathbb{P}}\left(\left|\int_0^{T_\rho} \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right| \geq \frac{\delta}{3} \mid \mu(B_{\mathbb{R}}(R) \times [0, T]) = 0\right) \\
& \leq \left(\frac{3}{\delta}\right)^2 \\
& \times \mathbb{E}^{\bar{\mathbb{P}}}\left[\left|\int_0^{T_\rho} \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right|^2 \mid \mu(B_{\mathbb{R}}(R) \times [0, T]) = 0\right].
\end{aligned}$$

Note that due to the fact that the random variables $\mu(B_{\mathbb{R}}(R) \times [0, T])$ and $\mu(A \times [0, T])$ for all $A \in \mathbb{R} \setminus B_{\mathbb{R}}(R)$ are independent, we get

$$\begin{aligned}
& \mathbb{E}^{\bar{\mathbb{P}}}\left[\left|\int_0^{T_\rho} \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right|^2 \mid \mu(B_{\mathbb{R}}(R) \times [0, T]) = 0\right] \\
& = \mathbb{E}^{\bar{\mathbb{P}}}\left[\left|\int_0^{T_\rho} \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right|^2\right].
\end{aligned}$$

The Burkholder inequality and the fact that $c(\theta(t, z)) \leq c(z)$ give

$$\begin{aligned} & \mathbb{E}^{\bar{\mathbb{P}}} \left[\left| \int_0^T \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z)) (\mu - \gamma)(dz, ds) \right|^2 \right] \\ & \leq \mathbb{E}^{\bar{\mathbb{P}}} \left[\int_0^T \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} \left| e^{-(t-s)A} Bc(\theta(s, z)) \right|^2 \lambda ds \right] \\ & \leq \mathbb{E}^{\bar{\mathbb{P}}} \left[\int_0^T \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} \left| e^{-(t-s)A} Bc(z) \right|^2 \lambda ds \right] \leq C T R^{1-\frac{2}{\alpha}}. \end{aligned}$$

Therefore, collecting all together

$$\mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|u^c(T, x, v) - u_\mu^\theta(T, x)| \leq \frac{\delta}{3}} \right] \geq \frac{1}{C(R)} \left(1 - \left(\frac{3}{\delta} \right)^2 C T R^{1-\frac{2}{\alpha}} \right).$$

Since R is chosen in such a way that

$$(31) \quad \left(\frac{3}{\delta} \right)^2 C T R^{1-\frac{2}{\alpha}} \leq \frac{1}{2}$$

and using the fact that

$$\bar{\mathbb{P}}(\mu(B_{\mathbb{R}}(R) \times [0, T]) = 0) = e^{-\lambda(B_{\mathbb{R}}(R))T}$$

we get

$$\begin{aligned} \mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|u_\mu^\theta(T, x)| \leq \delta} \right] & \geq \mathbb{E}^{\bar{\mathbb{P}}} \left[1_{|\Delta(T)| \leq \frac{\delta}{3}} \right] \geq \bar{\mathbb{P}}(\mu(B_{\mathbb{R}}(R) \times [0, T]) = 0) \\ & \quad \times \left(1 - \bar{\mathbb{P}} \left(\left| \int_0^T \int_{\mathbb{R} \setminus B_{\mathbb{R}}(R)} e^{-(t-s)A} Bc(\theta(s, z)) (\mu - \gamma)(dz, ds) \right| \geq \frac{\delta}{3} \mid \mu(B_{\mathbb{R}}(R) \times [0, T]) = 0 \right) \right) \\ & \geq e^{-\lambda(B_{\mathbb{R}}(R))T} C(R) \left(1 - \left(\frac{3}{\delta} \right)^2 C T R^{1-\frac{2}{\alpha}} \right) \geq \frac{1}{2} e^{-\lambda(B_{\mathbb{R}}(R))T} C(R). \end{aligned}$$

Hence, we have shown that

$$\bar{\mathbb{P}}(|u(T, x)| \leq \delta) \geq \frac{1}{2} e^{-\lambda(B_{\mathbb{R}}(R))T} C(R),$$

which leads to the assertion. \square

4. AN EXAMPLE - THE DAMPED WAVE EQUATION WITH BOUNDARY NOISE

As mentioned in the introduction, as example we consider an elastic string, fixed at one end and perturbed at the other end by a Lévy noise. Mathematically, the system can be formulated as damped wave equation with boundary Lévy noise.

Throughout this section suppose that we are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual condition. On this probability space we

assume that we are given a real valued Lévy process L . Let $T > 0$. We consider the system

$$(32) \quad \begin{cases} u_{tt}(t, \xi) - \Lambda u(t, \xi) + \alpha u_\xi(t, \xi) &= 0, & t \in (0, T), \xi \in (0, 2\pi), \\ u(t, 0) &= 0, & t \in (0, T), \\ u_\xi(t, 2\pi) &= \dot{L}_t, & t \in (0, T), \\ u(0, \xi) &= u_0(\xi), & u_t(0, \xi) = u_1(\xi), \end{cases}$$

where $\Lambda = \Delta$ the Laplacian and \dot{L} is the Radon Nikodym derivative of a real valued Lévy process with characteristic measure ν , $u_0 \in H_0^1(0, 2\pi)$ and $u_1 \in L^2(0, 2\pi)$.

Equation (32) can be reformulated as a evolution equation of order one. Henceforth, let us introduce the Hilbert space $\mathcal{H} = D(\Lambda^{\frac{1}{2}}) \times L^2(\mathcal{O})$ equipped with the scalar product

$$\langle w, z \rangle_{\mathcal{H}} = \langle \Lambda^{\frac{1}{2}} w_1, \Lambda^{\frac{1}{2}} z_1 \rangle + \langle w_1, w_2 \rangle, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{H} \text{ and } z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{H},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $L^2(\mathcal{O})$. Define an operator \mathcal{A} with domain $D(\mathcal{A}) = D(\Lambda) \times D(\Lambda^{\frac{1}{2}}) \rightarrow \mathcal{H}$ by

$$(33) \quad \mathcal{A} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

and $\mathcal{B}_\alpha : \mathcal{H} \rightarrow \mathcal{H}$ by

$$\mathcal{B}_\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha z_1 \end{pmatrix}.$$

It is not difficult to prove that \mathcal{A} generates a C_0 semigroup $(\mathcal{S}(t))_{t \geq 0}$ on \mathcal{H} . To be more precise, if $\{\lambda_n =: n \in \mathbb{N}\}$ are the eigenvalues and $\{\phi_n : n \in \mathbb{N}\}$ the eigenfunction of A , then $\{\mu_n : n \in \mathbb{R}\}$ with $\mu_n = \sqrt{|\lambda_n|}$, $\mu_{-n} = \mu_n$, $n \in \mathbb{N}$, are the eigenvalues and

$$\left\{ \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\mu_n} \phi_n \\ \phi_n \end{pmatrix} : n \in \mathbb{Z} \right\},$$

are the eigenfunction of \mathcal{A} (see [28, Proposition]). The semigroup \mathcal{S} can be written as

$$\mathcal{S}(t) \begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{R}} e^{\mu_n t} \left(\mu_n \left\langle \frac{df}{dx}, \frac{d\phi_n}{dx} \right\rangle_{L^2([0,1])} + \langle g, \phi_n \rangle_{L^2([0,1])} \right) \psi_n, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \mathbb{R}.$$

To rewrite (32) as a stochastic evolution equations on the Hilbert space \mathcal{H} we need to find a way of transforming the nonhomogeneous boundary conditions in (32) to homogeneous one. Therefore we introduce the operator $D_{B,\gamma}$. For every $a \in \mathbb{R}$, $v = D_{B,\gamma} a$ is a solution to the problem

$$\begin{cases} Av(\xi) &= \lambda v(\xi), & \xi \in \mathcal{O}, \\ v_\xi(2\pi) &= a, & v_\xi(0) = 0. \end{cases}$$

By a short calculation it follows that given $a \in \mathbb{R}$,

$$v = D_{B,1}(\xi) = \frac{a}{e^{2\pi} - e^{-2\pi}} (e^{-\xi} + e^\xi), \quad \xi \in [0, 2\pi].$$

Following the approach in [21] we see that (32) can be transformed to the following

$$(34) \quad \begin{cases} dX = (\mathcal{A} + \mathcal{B}_\alpha) X(t)dt + (\mathcal{A} - \mathcal{I}) \begin{pmatrix} 0 \\ D_{B,1}dL \end{pmatrix}, \\ X(0) = X_0, \end{cases}$$

where $X = (u, \dot{u})^T$, $X_0 = (u_0, u_1)^T$. Here u denotes the solution of (32). From now on we will work with (34).

First, note that by mimicking the proof of [21, Theorem 15.7.2] one can show that Problem (34) is well posed and, if $\int z^2 \nu(dz) < \infty$, then (34) has a unique mild solution which is a Markov-Feller process. In particular, the family of operators $(\mathcal{P}_t)_{t \geq 0}$ defined by

$$(35) \quad \mathcal{P}_t \phi(x) := \mathbb{E} \phi(X), \quad \phi \in C_b(\mathcal{H}), \quad t \geq 0.$$

is indeed a semigroup on $C_b(\mathcal{H})$. By means of Theorem 2.6 and Lemma 3.1 the following result can be obtained.

Theorem 4.1. *The Markovian semigroup $(\mathcal{P}_t)_{t \geq 0}$ defined by (35) has at most one invariant measure.*

We need to show some facts which are essential for the results in the previous sections to be applicable in for our example. First note, that the following system

$$(36) \quad \begin{cases} u_{tt}(t, \xi) - \Lambda u(t, \xi) + \alpha u(t, \xi) &= 0, \quad t \in (0, T), \xi \in (0, 2\pi), \\ u(t, 0) &= 0, \quad t \in (0, T), \\ u_\xi(t, 2\pi) &= v(t), \quad t \in (0, T), \\ u(0, \xi) &= u_0(\xi), \quad u_t(0, \xi) = u_1(\xi), \end{cases}$$

with control $v \in L^2([0, \infty); \mathbb{R})$ is approximate null controllable with vanishing energy, which is done in the following Lemma.

Lemma 4.2. *The system*

$$(37) \quad \begin{cases} \frac{\partial X(t)}{\partial t} = (\mathcal{A} + \mathcal{B}_\alpha) X(t) + (\mathcal{A} - \mathcal{I}) \begin{pmatrix} 0 \\ D_{B,1}v(t) \end{pmatrix}, \\ X(0) = X_0, \end{cases}$$

is approximate null controllable with vanishing energy.

Proof. It was proved in [9, Section 2.4] (see also e.g. [28, Example 11.2.6]) that (32) is exactly controllable at any time T , hence it is null controllable. Thanks to [9, Theorem 2.45] it is approximately null controllable. Now it remains to prove that it is approximate null controllable with vanishing energy. For this purpose we mainly follow the idea in [22]. Let us write \mathcal{H} as the direct sum $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$ where $\mathcal{H}_u = \{0\}$ and $\mathcal{H}_s = \mathcal{H}$. Therefore we see that [22, Hypothesis 1.1] are satisfied in our case. Moreover, since \mathcal{A} is the infinitesimal generator of a strongly continuous semigroup of contractions we can deduce from [19, Chapter 1, Corollary 3.6] that the spectrum $\sigma(\mathcal{A})$ is contained in $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\}$. This fact implies that $S(A) = \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(\mathcal{A})\} \leq 0$. Therefore we can deduce from [22, Theorem 1.1] that (37) is null controllable with vanishing energy. \square

Now we are ready to prove the existence and uniqueness of the invariant measure

Proof of Theorem 4.1. To show the existence of the invariant measure we can argue exactly as in Theorem (A.1).

It remains to show the uniqueness of the invariant measure. Owing to the Lemma 4.2 and Theorem 2.6 the semigroup \mathcal{P}_t is asymptotically strong Feller. By [13, Corollary 3.17] we know that if the semigroup is asymptotically strong Feller and there exists a point $x \in H$ such that $x \in \text{supp}(\mu)$, whenever ν is an invariant measure of $(\mathcal{P}_t)_{t \geq 0}$, then the Markovian $(\mathcal{P}_t)_{t \geq 0}$ semigroup admits almost one invariant measure. Therefore, we have to show that $x \in \text{supp}(\mu)$, i.e. for all $\delta > 0$, $\nu(\mathcal{B}_H(\delta)) > 0$. Since null controllability implies approximate null controllability, Lemma 3.1 can be applied and for all $\delta > 0$ and for all $x \in \bar{\mathcal{B}}_H(C)$ there exists a time $T_\delta > 0$ such that

$$(38) \quad u(T_\delta, x, \eta) \in \mathcal{B}_H(\delta).$$

It remains to show (20). In particular, we should check that there exists a constant $C > 0$ such that

$$(39) \quad \inf_{\{\mu \text{ is an invariant measure}\}} \mu(\bar{\mathcal{B}}(C)) > 0.$$

It follows that $0 \in \text{supp}(\mu)$ by the following observations. First, since μ is invariant we have

$$\mu(\mathcal{B}_H(\gamma)) \geq \mu(\bar{\mathcal{B}}_H(C)) \cdot \inf_{\{\mu \text{ is invariant measure}\}} \mu(\bar{\mathcal{B}}(C)).$$

Now, the estimates (39) and (21) give the assertion from which we easily complete the proof of the Theorem 4.1. \square

APPENDIX A. WELLPOSEDNESS OF THE MARKOVIAN SEMIGROUP AND EXISTENCE OF THE INVARIANT MEASURE

Let H be a Hilbert spaces, $A : H \rightarrow H$ be a generator of a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ on H and $B : \mathbb{R} \rightarrow D(A^{-\gamma})$ for some $\gamma < \frac{1}{2}$. (Let $\eta : \mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, \infty)) \rightarrow \mathbb{N}_0 \cup \{\infty\}$ be a Poisson random measure with intensity measure ν , where ν is a Lévy measure on \mathbb{R} . Let u the solution of the following stochastic evolution system

$$(40) \quad \begin{cases} du(t, x) &= Au(t, x) + \int_{\mathbb{R}} Bz \tilde{\eta}(dz, ds), \\ u(t, x) &= x \in H. \end{cases}$$

If $\int_{\mathbb{R}} |z|^2 \nu(dz) < \infty$, then the Markovian semigroup $(\mathcal{P}_t)_{t \geq 0}$ defined by (compare (6))

$$\mathcal{P}_t \phi(x) := \mathbb{E} \phi(u(x, t)), \quad \phi \in C_b(H), \quad t \geq 0,$$

is a semigroup on $C_b(H)$. That is $(\mathcal{P}_t)_{t \geq 0}$ satisfies (see [10])

- (1) $\mathcal{P}_t \circ \mathcal{P}_s = \mathcal{P}_{t+s}$;
- (2) for all $\phi \in C_b(H)$ and for all $x \in H$ we have $\lim_{t \rightarrow 0} \mathcal{P}_t \phi(x) = \phi(x)$.

Item (1) is clear. In order to verify (2) let $\phi \in C_b(H)$ with $|\phi|_\infty = 1$. Item (2) follows by the fact that $\lim_{t \rightarrow 0} \mathbb{E} \phi(u(t, x)) = \phi(x)$, or, for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\mathbb{E} \phi(u(t, x)) - \phi(x)| \leq \epsilon$ for all $0 \leq t < \delta$. Fix $\epsilon > 0$. Since ϕ is uniformly continuous on H , there exists a $\delta_1 > 0$ such that $|\phi(x) - \phi(y)| \leq \frac{\epsilon}{2}$ for all $x, y \in H$, $|x - y|_H \leq \delta_1$. Then for

$t \leq \delta := \frac{\epsilon}{6} \delta_1^2$ we know by the Chebyscheff inequality that $\mathbb{P}(|u(t, x) - x|_H \geq \delta_1) \leq \frac{\epsilon}{2}$. Hence,

$$\begin{aligned} \mathbb{E}|\phi(u(t, x)) - \phi(x)| &\leq \\ &+ \mathbb{E}[|\phi(u(t, x)) - \phi(x)| \mid |u(t, x) - x| \geq \delta_1] \mathbb{P}(|u(t, x) - x|_H \geq \delta_1) \\ &+ \mathbb{E}[|\phi(u(t, x)) - \phi(x)| \mid |u(t, x) - x| < \delta_1] \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

It follows that $(\mathcal{P}_t)_{t \geq 0}$ on $C_b(H)$ is, in fact, a semigroup.

We are interested under which condition $(\mathcal{P}_t)_{t \geq 0}$ admits an invariant measure. Here, the existence of the invariant measure can be shown by an application of the Krylov-Bogoliubov Theorem (see [11, Theorem 3.1.1],). First, we will define for $T > 0$ and $x \in H$ the following probability measure on $(H, \mathcal{B}(H))$

$$(41) \quad \mathcal{B}(H) \ni \Gamma \mapsto R_T(x, \Gamma) := \frac{1}{T} \int_0^T P_t(x, \Gamma) dt.$$

In addition, for any $\sigma \in M_1(H)$, let $R_T^* \sigma$ be defined by

$$\mathcal{B}(H) \ni \Gamma \mapsto \int_H R_T(x, \Gamma) \sigma(dx).$$

Corollary 3.1.2 in [11] says, that if for some probability measure σ on $(H, \mathcal{B}(H))$ and some sequence $T_n \uparrow \infty$, the sequence $\{R_{T_n}^* \sigma : n \in \mathbb{N}\}$ is tight, then there exists an invariant measure for $(\mathcal{P}_t)_{t \geq 0}$. That means, it is sufficient to show that for all $\epsilon > 0$ for all $x \in H$, there exists a compactly embedded subspace $E \hookrightarrow H$, a $R > 0$ such that we have for all $T > 0$

$$(42) \quad \frac{1}{T} \int_0^T \mathbb{P}(|u(t, x)|_E \geq R) dt < \epsilon.$$

However, if there exists a constant $C > 0$ and a number $p > 0$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}|u(t, x)|_E^p \leq C, \quad T \geq 0,$$

then (42) holds.

Observe, if A generates a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ of contraction on H , then

$$\int_0^\infty |e^{-tA}|_{L(H, D(A^{-\gamma}))}^2 dt < \infty.$$

Theorem A.1. *Let H_1 be a Hilbert space such that $H_1 \hookrightarrow H$ compactly, Assume A generates a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ of contraction on H and H_1 , and*

$$\int_0^\infty |S(t-s)B|_{L(D(A^{-\gamma}, H_1))}^2 ds < \infty,$$

Then, if

$$\int_{\mathbb{R}} |z|^2 \nu(dz) < \infty,$$

the Markovian semigroup $(\mathcal{P}_t)_{t \geq 0}$ admits an invariant measure.

Proof. First, Equation (1) can be written as follows

$$\begin{cases} du(t, x) &= Au(t, x) dt + \int_{\mathcal{Z}} B z \tilde{\eta}(dz, ds), \quad t > 0, \\ u(0, x) &= x. \end{cases}$$

If A is a semigroup of contractions then (see [15])

$$\int_0^\infty \int_{\mathcal{Z}} |e^{-tA} Bz|^2 \nu(dz) dt < \infty$$

Now, we will show that there exists a constant $C > 0$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E} |u(t, x)|_{H_1}^2 \leq C, \quad T \geq 0.$$

Due to standard arguments (see [15]) we get

$$\begin{aligned} \mathbb{E} |u(t, x)|_{H_1}^2 &\leq C_1 |e^{-tA} x|_{H_1}^2 + C_2 \mathbb{E} \left| \int_0^t e^{-(t-s)A} B z \nu(dz) ds \right|_{H_1}^2 \\ &\leq C_1 |e^{-tA} x_0|_{H_1}^2 + C_2 \mathbb{E} \int_0^t \int_{\mathbb{R}} |e^{-(t-s)A} Bz|_{H_1}^2 \nu(dz) ds. \end{aligned}$$

Due to the assumption, the first and the second summand are bounded uniformly for all $t \geq 0$. Hence, there exists a constant $C > 0$ such that

$$\sup_{0 \leq t < \infty} \mathbb{E} |u(t, x)|_{H_1}^2 \leq C.$$

Now, the Chebyshev inequality leads for any $R > 0$ to

$$\begin{aligned} \frac{1}{T} \int_0^T \mathbb{P}(|u(t, x)|_{H_1} \geq R) dt &\leq \frac{1}{T} \int_0^T \frac{\mathbb{E} |u(t, x)|_{H_1}^2}{R^2} ds \\ \frac{1}{T} \int_0^T \frac{\mathbb{E} |u(t, x)|_{H_1}^2}{R^2} ds &\leq \frac{C}{R^2}. \end{aligned}$$

Given $\epsilon > 0$ and taking $R > (\frac{C}{\epsilon})^{\frac{1}{2}}$, inequality (42) follows. \square

APPENDIX B. TECHNICAL PRELIMINARIES

In this section we will show that one can find a transformation $\theta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that for any $v \in \mathbb{R}$,

$$\int_{\mathbb{R}} z(\nu_\theta - \nu)(dz) = v,$$

and ν_θ is a Lévy measure. Here

$$\nu_\theta : \mathcal{B}(\mathbb{R}) \ni B \mapsto \int_{\mathbb{R}} 1_B(\theta(v, z)) \nu(dz).$$

In order to find the transformation, it is convenient to switch the representation of the Poisson random measure as in the beginning of the proof of Theorem 2.6. Let ν be a Lévy measure satisfying Hypothesis ?? and 1. Let

$$c : \mathbb{R}^+ \ni r \mapsto \sup_{\rho > 0} \left\{ \int_\rho^\infty k(s) ds \geq r \right\}.$$

To analyse the effect of the perturbation, we define a function w by

(43)

$$[0, \infty) \ni K \mapsto w(K) := \int_{\mathbb{R}^+} (c(z) - c(z + \rho(K, z))) dz \in \mathbb{R},$$

where $\rho : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$(44) \quad \rho(K, z) := \begin{cases} Kz^{-\beta_1} & z \in (K^{\gamma_1}r_1, K^{\gamma_1}2r_1) \text{ and } K \geq 1, \\ 0 & z \notin (K^{\gamma_1}r_1 - \frac{1}{4}, K^{\gamma_1}2r_1 + \frac{1}{4}) \text{ and } K \geq 1, \\ Cz^{-\beta_2} & z \in (r_1, r_1(1 + K^{\gamma_2})) \text{ and } K < 1, \\ 0 & z \notin (r_1 - \frac{1}{4}, r_1(1 + K^{\gamma_2}) + \frac{1}{4}) \text{ and } K < 1, \\ \text{differentiable interpolated elsewhere} \end{cases}$$

and

$$\beta_1 = \frac{3 - 2\alpha}{\alpha(3\alpha - 5)}, \quad \beta_2 = -1, ,$$

and

$$\gamma_1 = 5\alpha - 3\alpha^2, \quad \gamma_2 = -\frac{\alpha}{2}.$$

The constant $C > 0$ has to be chosen in such a way, that $K \mapsto w(K)$ is continuously in K . Moreover, let

$$(45) \quad w : \mathbb{R}_0^+ \ni x \mapsto w(x).$$

Lemma B.1. *The function $w : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is invertible. In particular, there exists a constant $C = C(r_1) > 0$ such that*

$$w^{-1}(x) \leq C \left(1 + |x|^{\frac{(\beta+1)\alpha}{\alpha-1}} \right).$$

Proof. We will show that there exists a function $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $K(w(x)) = w(K(x)) = x$ for all $x \in \mathbb{R}^+$ with $w(x) > 0$. Moreover, there exists a constant $C > 0$ such that

$$(46) \quad K(x) \leq C \left(1 + |x|^{\frac{(\beta+1)\alpha}{\alpha-1}} \right).$$

Hence, we have to show that $w(K) \in \mathbb{R}^+$ defined by

$$(47) \quad \mathbb{R}^+ \ni K \mapsto w(K) = \int_0^\infty [c(z) - c(z + \rho(K, z))] dz$$

is invertible. We start by verifying the following properties

- (1) $w(K) \in \mathbb{R}^+$;
- (2) the function $\mathbb{R}^+ \ni K \mapsto f(K) \in \mathbb{R}^+$ is continuous.
- (3) the function $\mathbb{R}^+ \ni K \mapsto f(K) \in \mathbb{R}^+$ is injective.
- (4) the function $\mathbb{R}^+ \ni K \mapsto f(K) \in \mathbb{R}^+$ is surjective.

It follows, in particular, from (2), (3) and (4), that the function f is invertible.

In fact, (1) follows since ν is selfdecomposable, and hence, c is also selfdecomposable. In order to show Item (2) we take into account that the function $\mathbb{R}^+ \ni K \mapsto w(K) \in \mathbb{R}^+$ is strictly decreasing and continuous. In order to show Item (3) we will show, that $\lim_{K \rightarrow \infty} w(K) = \infty$. Since $w(0) = 0$ and w is continuous in 1, the claim follows.

Firstly, we consider the case $K > 1$. Here we have

$$\begin{aligned} w(K) &= \int_0^\infty [c(z) - c(z + \rho(K, z))] dz = \int_{r_1 K^{\gamma_1}}^{K^{\gamma_1} 2r_1} \int_z^{z+\rho(K, z)} \frac{d}{dy} c(y) dy dz \\ &= \int_{K^{\gamma_1} r_1}^{K^{\gamma_1} 2r_1} \int_z^{z+K^{\gamma_1} \rho(K, z)} \frac{d}{dy} c(y) dy dz \geq \int_{K^{\gamma_1} r_1}^{K^{\gamma_1} 2r_1} \rho(K, z) \frac{d}{dy} c(z + \rho(K, z)) dz. \end{aligned}$$

Hypothesis ?? and 1 give for $\tilde{\gamma}_1 = \gamma_1(1 + \beta_1) - 1 = \frac{17(\frac{\alpha-12}{17\alpha}+1)\alpha}{3\alpha-2} - 1 = 5$

$$\begin{aligned}
\dots &\geq \delta_0 K \int_{r_1 K^{\gamma_1}}^{K^{\gamma_1} 2r_1} \frac{z^{-\beta_1}}{\left(z + \frac{K}{z^{\beta_1}}\right)^{\frac{1}{\alpha}+1}} dz \geq \delta_0 K \int_{r_1 K^{\gamma_1}}^{K^{\gamma_1} 2r_1} \frac{z^{-\beta_1+\beta_1(\frac{1}{\alpha}+1)}}{(z^{1+\beta_1} + K)^{\frac{1}{\alpha}+1}} dz \\
&= \delta_0 K K^{-\frac{1+\alpha}{\alpha}} \int_{r_1 K^{\gamma_1}}^{K^{\gamma_1} 2r_1} \frac{z^{\frac{\beta_1}{\alpha}}}{\left(\frac{z^{1+\beta_1}}{K} + 1\right)^{\frac{1}{\alpha}+1}} dz \\
&= K^{-\frac{1}{\alpha}} \int_{r_1^{1+\beta_1} K^{\tilde{\gamma}_1}}^{(2r_1)^{1+\beta_1} K^{\tilde{\gamma}_1}} \frac{(Ku)^{\frac{1}{1+\beta_1} \frac{\beta_1}{\alpha}}}{(u+1)^{\frac{1}{\alpha}+1}} K^{\frac{1}{\beta_1+1}} u^{\frac{1}{\beta_1+1}-1} du \\
&= K^{\frac{-(1+\beta_1)+\beta_1+\alpha}{\alpha(1+\beta_1)}} \int_{r_1^{1+\beta_1} K^{\tilde{\gamma}_1}}^{(2r_1)^{1+\beta_1} K^{\tilde{\gamma}_1}} \frac{u^{\frac{\beta_1+\alpha-(\beta_1+1)\alpha}{(\beta_1+1)\alpha}}}{(u+1)^{\frac{1}{\alpha}+1}} du \\
&= K^{\frac{\alpha-1}{\alpha(1+\beta_1)}} \int_{r_1^{1+\beta_1} K^{\tilde{\gamma}_1}}^{(2r_1)^{1+\beta_1} K^{\tilde{\gamma}_1}} \frac{u^{\frac{\beta_1(1-\alpha)}{(\beta_1+1)\alpha}}}{(u+1)^{\frac{1}{\alpha}+1}} du.
\end{aligned}$$

First, note that we have $\tilde{\gamma}_1 = -3\alpha^2 + 7\alpha - 4 < 0$ for $1 < \alpha \leq 2$ and $(1 + \beta_1) = \frac{6(3\alpha-2)}{17\alpha} > 0$. That means, we have for all $u \geq r_1^{1+\beta_1} K^{\tilde{\gamma}_1}$

$$\frac{1}{(1+u)^{\frac{1}{\alpha}+1}} \geq \frac{(r_1^{(1+\beta_1)} K^{\tilde{\gamma}_1})^{\frac{1+\alpha}{\alpha}}}{(1+r_1^{(1+\beta_1)} K^{\tilde{\gamma}_1})^{\frac{1}{\alpha}+1}} u^{-(\frac{1}{\alpha}+1)} \geq u^{-(\frac{1}{\alpha}+1)}.$$

Integration and substituting of β_1 give for $K > 1$

$$\begin{aligned}
w &= \int_0^\infty [c(z) - c(z + K\rho(z))] dz \\
&\geq K^{\frac{\alpha-1}{\alpha(1+\beta_1)}} \frac{r_1^{(1+\beta_1)\frac{1+\alpha}{\alpha}}}{(1+r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}} \int_{r_1^{1+\beta_1} K^{\tilde{\gamma}_1}}^{(2r_1)^{1+\beta_1} K^{\tilde{\gamma}_1}} u^{\frac{\beta_1(1-\alpha)}{(\beta_1+1)\alpha} - \frac{1}{\alpha} - 1} du \\
&= K^{\frac{\alpha-1}{\alpha(1+\beta_1)}} \frac{r_1^{(1+\beta_1)\frac{1+\alpha}{\alpha}}}{(1+r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}} \int_{r_1^{1+\beta_1} K^{\tilde{\gamma}_1}}^{(2r_1)^{1+\beta_1} K^{\tilde{\gamma}_1}} u^{-\frac{\alpha\beta_1+1}{(\beta_1+1)\alpha} - 1} du \\
&= C K^{\frac{\alpha-1}{\alpha(1+\beta_1)} + \tilde{\gamma}_1 \frac{1+\alpha}{\alpha}} \frac{r_1^{(1+\beta_1)\frac{1+\alpha}{\alpha}}}{(1+r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}} r_1^{-(1+\beta_1)\frac{\alpha\beta_1+1}{(\beta_1+1)\alpha}} \left(C(2K)^{-\frac{\tilde{\gamma}_1(\alpha\beta_1+1)}{\alpha(1+\beta_1)}} \right) \\
&\geq C K^{\frac{\alpha-1-\tilde{\gamma}_1(\alpha\beta_1+1)}{\alpha(1+\beta_1)}} \frac{r_1^{\frac{\beta_1+\alpha}{\alpha}}}{(1+r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}} \\
&\geq C K^{\alpha-1} \frac{r_1^{\frac{\beta_1+\alpha}{\alpha}}}{(1+r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}}.
\end{aligned}$$

It follows that

$$(48) \quad K \leq r_1^{\frac{\alpha\beta_1+1}{\alpha}} w^{\frac{1}{\alpha-1}},$$

and, therefore

$$\lim_{K \rightarrow \infty} \int_{r_1}^{\infty} [c(z) - c(z + \rho(K, z))] dz = \infty.$$

From (1), (2) and (3) it follows that the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by (47) is invertible. For any $z \in \mathbb{R}^+$ let us write $K(z) = K$ iff $f(K) = z$. Inequality (48) implies inequality (46).

It remains to investigate the rate of grow for $0 < K \leq 1$. Here, we have

$$\begin{aligned} & \int_0^{\infty} [c(z) - c(z + \rho(K, z))] dz \\ &= \int_{r_1}^{r_1(1+K)^{\gamma_2}} \rho(K, z) \frac{d}{dz} c(z + \rho(K, z)) dz du, \end{aligned}$$

which implies the existence of a positive constant $C(r_1, \alpha, \gamma_2, \beta_2)$ such that

$$\begin{aligned} & \int_0^{\infty} [c(z) - c(z + \rho(K, z))] dz \\ &= C(r_1, \alpha, \gamma_2, \beta_2) \int_{r_1}^{r_1(1+K)^{\gamma_2}} \frac{z^{-\beta_2}}{(z + z^{-\beta_2})^{1+\frac{1}{\alpha}}} dz \\ &= C(r_1, \alpha, \gamma_2, \beta_2) \int_{r_1}^{r_1(1+K)^{\gamma_2}} \frac{z^{-\beta_2 \frac{\alpha-1}{\alpha}}}{(z^{1+\beta_2} + 1)^{1+\frac{1}{\alpha}}} dz. \end{aligned}$$

By changing of variables we get that

$$\begin{aligned} & \int_0^{\infty} [c(z) - c(z + \rho(K, z))] dz \\ &\geq C(r_1, \alpha, \gamma_2, \beta_2) \int_{r_1^{1+\beta_2}}^{r_1^{1+\beta_2}(1+K^{\gamma_2})^{(1+\beta_2)}} \frac{u^{-\frac{\beta_2(\alpha-1)}{\alpha(\beta_2+1)}}}{(u+1)^{1+\frac{1}{\alpha}}} u^{\frac{\beta_2}{1+\beta_2}} du \\ &\geq C(r_1, \alpha, \gamma_2, \beta_2) \int_{r_1^{1+\beta_2}}^{r_1^{1+\beta_2}(1+K^{\gamma_2})^{(1+\beta_2)}} \frac{u^{\frac{\beta_2}{\alpha(\beta_2+1)}}}{(u+1)^{1+\frac{1}{\alpha}}} du. \end{aligned}$$

Since $r_1^{1+\beta_2} \leq u \leq r_1^{1+\beta_2}(1+K^{\gamma_2})^{(1+\beta_2)}$, we get

$$\begin{aligned} & \int_0^{\infty} [c(z) - c(z + \rho(K, z))] dz \\ &\geq C(r_1, \alpha, \gamma_2, \beta_2) \frac{1}{(1+r_1^{1+\beta_2}(1+K^{\gamma_2})^{(1+\beta_2)})^{1+\frac{1}{\alpha}}} \int_{r_1^{1+\beta_2}}^{[r_1(1+K^{\gamma_2})]^{(1+\beta_2)}} u^{-\frac{1+\alpha+\alpha\beta_2}{\alpha(\beta_2+1)}} du, \end{aligned}$$

which implies that

$$\begin{aligned} & \int_0^{\infty} [c(z) - c(z + \rho(K, z))] dz \\ &\geq C(r_1, \alpha, \gamma_2, \beta_2) \frac{r_1^{-\frac{1}{\alpha}}}{(1+[r_1(1+K^{\gamma_2})]^{\beta_2+1})^{1+\frac{1}{\alpha}}} K^{-\gamma_2 \frac{1+\alpha+\alpha\beta_2}{\alpha}} \\ &\geq C(r_1, \alpha, \gamma_2, \beta_2) K^{-\gamma_2 \frac{1+\alpha+\alpha\beta_2}{\alpha}} \geq C(r_1, \alpha, \gamma_2, \beta_2) K^{\frac{1}{2}}. \end{aligned}$$

This proves of the following two corollaries. \square

Corollary B.2. *Under Hypothesis ?? for any $\tilde{r} > r_1$ and $v \in \mathbb{R}_0^+$ there exists a number K such that*

$$\int_{\tilde{r}}^{\infty} [c(r) - c(\theta(K, r))] dr = v.$$

Moreover, there exists a constant $C(\tilde{r}) > 0$ such that

$$\int_{\tilde{r}}^{\infty} |\rho_z(|K|, r)| dr \leq C(\tilde{r}) |v|^2.$$

Corollary B.3. *Under Hypothesis ?? for any $\tilde{r} > r_1$ and $v \in \mathbb{R}$ there exists a transformation $\theta : \mathbb{R} \rightarrow \mathbb{R}$ given by*

$$\theta(v, z) := \begin{cases} z + \rho(|v|, z) & \text{if } v \geq 0, \\ -z - \rho(|v|, z) & \text{if } v < 0, \end{cases}$$

such that

$$\int_{\tilde{r}}^{\infty} [c(z) - c(\theta(v, z))] dz = v.$$

Moreover, there exists a constant $C(\tilde{r}) > 0$ such that

$$\int_{\tilde{r}}^{\infty} |\rho_z(|K|, z)| dz \leq C(\tilde{r}) |v|^2.$$

APPENDIX C. CHANGE OF MEASURE FORMULA

Let μ be a Poisson random measure over $\mathfrak{A} = (\Omega, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F})$ with compensator $\gamma = \lambda \cdot \lambda$. Let $c : \mathbb{R} \rightarrow \mathbb{R}$ the transformation defined by (9). Let $g \in L^2([0, \infty); \mathbb{R})$ be a predictable process and let

$$(49) \quad \psi : [0, \infty) \times \mathbb{R} \ni (s, z) \mapsto z + g(s)\rho(z) \in \mathbb{R}.$$

Combining Corollary B.3 and Example 1.9 of [14] one can verify the following Lemma.

Lemma C.1. *There exists a probability measure \mathbb{Q}^ψ on \mathfrak{A} such that the Poisson random measure μ_ψ defined by*

$$\mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, \infty)) \ni A \times I \mapsto \int_I \int_{\mathbb{R}} 1_A(\psi(s, z)) \mu(dz, ds)$$

has compensator γ . For $t \geq 0$ let \mathbb{Q}_t^ψ , respectively, \mathbb{P}_t , be the projection of \mathbb{Q}^ψ onto \mathcal{F}_t , respectively, of \mathbb{P} onto \mathcal{F}_t . Then the density process given by

$$[0, \infty) \ni t \mapsto \mathcal{G}(t) := \frac{d\mathbb{Q}_t^\psi}{d\mathbb{P}_t}, \quad t > 0,$$

satisfy

$$\begin{cases} d\mathcal{G}(t) &= \mathcal{G}(t-) \int_{\mathbb{R}} (1 - \psi_z(z)) (\mu - \gamma)(dz, dt), \\ &= \mathcal{G}(t-) \int_{\mathbb{R}} g(s)\rho_z(z) (\mu - \gamma)(dz, dt), \\ \mathcal{G}(0) &= 1, \end{cases}$$

where $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by (44) and ρ_z denotes the derivative of ρ .

Proof. The proof is done via the Laplace transform. Let $\xi = \{\xi(t) : 0 \leq t < \infty\}$ be given by

$$\begin{cases} d\xi(t) &= \int_{\mathbb{R}} c(z)(\mu_{\psi} - \gamma)(dz, dt), \\ \xi(0) &= 0. \end{cases}$$

Then under \mathbb{Q}^{ψ} the Laplace transform is given by

$$\mathbb{E}^{\mathbb{Q}^{\psi}} e^{-\lambda \xi(t)} = e^{\int_{\mathbb{R}} [e^{-\lambda c(z)} - 1 + \lambda c(z)] \lambda(dz)}$$

Rewriting ξ gives

$$\begin{cases} d\xi(t) &= \int_{\mathbb{R}} c(\psi(t, z))(\mu - \gamma)(dz, dt) + \int_{\mathbb{R}} [c(\psi(s, z)) - c(z)] \gamma(dz, dt), \\ \xi(0) &= x_0. \end{cases}$$

Let $M_{\lambda} = \{M_{\lambda}(t) : 0 \leq t < \infty\}$ be given by $M_{\lambda}(t) = e^{-\lambda \xi(t)}$, $0 \leq t < \infty$. Now, we will show that $\mathbb{E}^{\mathbb{P}} M_{\lambda}(t) \mathcal{G}(t) = \mathbb{E}^{\mathbb{Q}^{\psi}} e^{-\lambda \xi(t)}$. First $M_{\lambda}(t)$ solves

$$\begin{aligned} dM_{\lambda}(t) &= -\lambda \int_{\mathbb{R}} M_{\lambda}(t-) [c(\psi(t, z)) - c(z)] \gamma(dz, dt) \\ &\quad + \int_{\mathbb{R}} M_{\lambda}(t-) [e^{-\lambda c(\psi(t, z))} - 1] (\mu - \gamma)(dz, dt) \\ &\quad + \int_{\mathbb{R}} M_{\lambda}(t-) [e^{-\lambda c(\psi(t, z))} - 1 + \lambda c(\psi(t, z))] \gamma(dz, dt), \\ M_{\lambda}(0) &= 1. \end{aligned}$$

Therefore, $\mathcal{Z}_{\lambda}(t) = M_{\lambda}(t) \mathcal{G}(t)$ is given by

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \mathcal{Z}_{\lambda}(t) &= -\lambda \mathbb{E}^{\mathbb{P}} \int_0^t \int_{\mathbb{R}} \mathcal{Z}_{\lambda}(s-) [c(\psi(s, z)) - c(z)] \lambda(dz) ds \\ &\quad + \mathbb{E}^{\mathbb{P}} \int_0^t \int_{\mathbb{R}} \mathcal{Z}_{\lambda}(s-) [e^{-\lambda c(\psi(s, z))} - 1 + \lambda c(\psi(s, z))] \lambda(dz) ds \\ &\quad + \mathbb{E}^{\mathbb{P}} \int_0^t \mathcal{Z}_{\lambda}(s-) \int_{\mathbb{R}} [e^{-\lambda c(\psi(s, z))} - 1] [\psi_z(s, z) - 1] \lambda(dz) ds \\ &= \mathbb{E}^{\mathbb{P}} \int_0^t \int_{\mathbb{R}} \mathcal{Z}_{\lambda}(s-) [\lambda c(z) - \lambda c(\psi(s, z)) + e^{-\lambda c(\psi(s, z))} - 1 \\ &\quad + \lambda c(\psi(s, z)) + e^{-\lambda c(\psi(s, z))} \psi_z(s, z) - \psi_z(s, z) - e^{-\lambda c(\psi(s, z))} + 1] \gamma(dz, ds) \\ &= \mathbb{E}^{\mathbb{P}} \int_0^t \int_{\mathbb{R}} \mathcal{Z}_{\lambda}(s-) [e^{-\lambda c(\psi(s, z))} \psi_z(s, z) - \psi_z(s, z) + \lambda c(z)] \gamma(dz, ds). \\ &= \mathbb{E}^{\mathbb{P}} \int_0^t \int_{\mathbb{R}} \mathcal{Z}_{\lambda}(s-) [e^{-\lambda c(\psi(s, z))} - 1] \psi_z(s, z) \lambda(dz) ds \\ &\quad + \lambda \mathbb{E}^{\mathbb{P}} \int_0^t \int_{\mathbb{R}} \mathcal{Z}_{\lambda}(s-) c(z) \gamma(dz, ds). \end{aligned}$$

Substitution gives

$$\mathbb{E}^{\mathbb{P}} \mathcal{Z}_{\lambda}(t) = \mathbb{E}^{\mathbb{P}} \int_0^t \int_{\mathbb{R}} \mathcal{Z}_{\lambda}(s-) [e^{-\lambda c(z)} - 1 + \lambda c(z)] \gamma(dz, ds).$$

Since

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_\psi} \left[e^{-\lambda \xi(t)} \right] &= \mathbb{E}^{\mathbb{P}} \left[\mathcal{G}(t) e^{-\lambda \xi(t)} \right] = \mathbb{E}^{\mathbb{P}} [\mathcal{Z}_\lambda(t)] \\ &= \exp \left(\int_0^t \int_{\mathbb{R}} \left[e^{-\lambda c(z)} - 1 + \lambda c(z) \right] \gamma(dz, dt) \right), \end{aligned}$$

the Proposition follows. \square

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